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Spherical CR-Manifolds of Dimension 3

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Abstract. A spherical CR-structure on a smooth $(2n-1)$-manifold M is a maximal collection of distinguished charts modeled on the boundary $\partial H^n_{\mathbb{C}}$ of the complex hyperbolic space, where coordinate changes are restrictions of transformations from $\mathrm{PU}(n,1)$. There exists a development map $d\colon \widetilde{M} \to \partial H^n_{\mathbb{C}}$, where $\widetilde{M}$ is the universal covering of M, which is a local diffeomorphism. We study properties of the development maps and holonomy groups of spherical CR-structures on compact 3-dimensional manifolds. We also give constructions of fundamental domains for some discrete subgroups of $\mathrm{PU}(2,1)$.

1. Introduction

A geometrical structure on a real analytic n-manifold M is a maximal collection of charts modeled on an n-dimensional homogeneous space X of a Lie group G whose coordinate changes are restrictions of transformations from G. We call such a structure an (X,G)-structure. We say in this case the manifold M is an (X,G)-manifold or is modeled on (X,G). Important examples of these structures include all locally homogeneous Riemannian structures as well as conformally flat, affinely flat, projectively flat and spherical CR-structures. Recall that a geometrical structure on a n-manifold M is called conformally flat if $X = S^n$ and $G = \mathrm{SO}(n+1,1)$ is the group of conformal transformations of X, where $\dim X \geq 2$. A geometrical structure is called a spherical CR-structure if $X = S^{2n-1}$ is the boundary of the unit ball in $\mathbb{C}^n$ and $G = \mathrm{PU}(n,1)$ is the group of biholomorphisms of the unit ball acting on its boundary by CR-automorphisms. The analogy between both structures is clear as conformally flat structures are modeled on the boundary of real hyperbolic space and spherical CR-structures are modeled on the boundary

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of complex hyperbolic space.

Besides the many parallels between spherical CR-geometry and conformally flat geometry, they have important differences. For instance, it is known that the 3-dimensional torus has a conformally flat structure, but it has no spherical CR-structure. Another interesting example is a 2-torus bundle over the circle. This manifold admits a conformally flat structure if and only if the attaching map of the bundle is periodic and it admits a spherical CR-structure if and only if its attaching map $A \in \mathrm{SL}(2, \mathbb{Z})$ has infinite order, but all its eigenvalues are ± 1 [**G1**]. There are also very interesting open questions. One knows that hyperbolic 3-manifolds are conformally flat, but we do not know any example of spherical CR-structure of any member of this class of manifolds. Similarly, trivial and some nontrivial circle bundles over a surface S of genus $g \geq 2$ have conformally flat structures [**GLT, Ka**] and some Seifert fiber spaces have spherical CR-structures [**BS**], but we know nothing about the existence of spherical CR-structures on the trivial circle bundle over S.

One of the main tools in the study of a geometrical structure is the development map and its holonomy homomorphism. The purpose of this paper is to study the development map of 3-dimensional closed spherical CR-manifolds.

Recall now the notions of development map and holonomy of an (X, G)-manifold. For details and proofs, see [**KP, Ku, T**].

Development theorem. *Let M be an (X, G)-manifold, and let $p\colon \widetilde{M} \to M$ denote the universal covering of M with covering group $\pi_1(M)$. Then there exists a pair (d, d^*), where $d\colon \widetilde{M} \to X$ is an (X, G)-local diffeomorphism, and $d^*\colon \pi_1(M) \to G$ is a homomorphism satisfying the equivariance condition*

$$d \circ \gamma = d^*(\gamma) \circ d$$

for all $\gamma \in \pi_1(M)$.

The map d is called a development map for the (X, G)-structure. The homomorphism d^* is called the holonomy homomorphism, and the group $\Gamma^* = d^*(\pi_1(M))$ is called the holonomy group for the (X, G)-structure.

A pair (d, d^*) is called a *development* pair and is a useful globalization of an (X, G)-structure defined by local coordinates. The development map pulls back the (X, G)-structure from X to $\widetilde{M}$ and thus defines an (X, G)-structure on $\widetilde{M}$. The holonomy homomorphism d^* determines the action of $\pi_1(M)$ on $\widetilde{M}$ by (X, G)-automorphisms. Thus, a development pair completely determines the (X, G)-structure on M. Moreover, if $(\tilde{d}, \tilde{d}^*)$ is another pair for the same (X, G)-structure, then there exists $h \in G$ such that $d = h \circ \tilde{d}$ and $\tilde{d}^*(\gamma) = h \circ d(\gamma) \circ h^{-1}$ for all $\gamma \in \pi_1(M)$.

There are some results describing development pairs for general (X, G)-structures, but the most complete picture has been obtained only for conformally flat structures.

As for spherical CR-structures, we know only few results in this direction. First, we notice that the spherical homogeneous CR-manifolds have been classified by Burns and Shnider [**BS**]. Recently Miner [**M**] has classified spherical CR-structures on closed manifolds with amenable holonomy group. Finally, Kamishima and Tsuboi have obtained a classification of closed spherical CR-manifolds admitting nontrivial CR-vector fields [**KT**].

The main results of this paper are the following.

Theorem 3.1. *Let M be a closed 3-dimensional spherical CR-manifold with infinite fundamental group. Then the following conditions are equivalent:*

a) $d(\widetilde{M}) = D \neq S^3$,

b) *$d: \widetilde{M} \to D$ is a covering map,*

c) *The holonomy group $\Gamma^* = d^*(\pi_1(M))$ acts discontinuously on D.*

We will call geometric circles on the boundary of the complex hyperbolic space $H^2_{\mathbb{C}}$ the intersections of S^3 with the boundaries of totally geodesic submanifolds of real dimension 2 in $H^2_{\mathbb{C}}$.

A spherical CR-structure on a 3-manifold M will be called special if the holonomy group Γ^* leaves invariant a geometric circle in S^3.

Theorem 3.2. *Let M be a closed 3-manifold with a special spherical CR-structure. Suppose that $\pi_1(M)$ is infinite. Then d is not surjective and the holonomy group Γ^* is discrete.*

These results show the difference between conformally flat and spherical CR-structures on closed 3-dimensional manifolds, see, for instance, [**Ka**], [**K**], [**GKam1**], [**GKam2**], [**GK**].

As noted by Goldman [**G2**], in general, the development maps of conformally flat structures on closed 3-dimensional manifolds fail to be covering maps onto their images. Using the operation of connected sums of spherical CR-structures, [**BS**], [**F**], we construct spherical CR-structures on closed 3-dimensional manifolds whose development maps are surjective but not covering onto their images.

Finally, we construct explicit fundamental domains for some discrete subgroups of $\mathrm{PU}(2,1)$.

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2. Preliminaries

2.1 Complex hyperbolic space and its boundary

2.1.1. Let C^{n+2} denote the complex vector space, equipped with the Hermitian form

$$b(z,w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

Consider the following subspaces in C^{n+2}:

$$V_0 = \{z \in \mathbb{C}^{n+2} : b(z,z) = 0\}$$
$$V = \{z \in \mathbb{C}^{n+2} : b(z,z) < 0\}$$

Let $P: \mathbb{C}^{n+2} \setminus \{0\} \to \mathbb{C}P^{n+1}$ be the canonical projection onto the complex projective space. Then $H^{n+1}_{\mathbb{C}} = P(V)$ equipped with the Bergman metric is the complex hyperbolic space. The orientation preserving isometry group of $H^{n+1}_{\mathbb{C}}$ is $\mathrm{PU}(n+1,1)$ acting by linear projective transformations. Also, $\mathrm{PU}(n+1,1)$ is the group of biholomorphic transformations of $H^{n+1}_{\mathbb{C}}$.

Put $S^{2n+1} = P(V_0)$. Then S^{2n+1} is the boundary of $H^{n+1}_{\mathbb{C}}$. We may consider $H^{n+1}_{\mathbb{C}}$ and S^{2n+1} as the unit ball and the unit sphere in

C^{n+1}. The group of CR-automorphisms of S^{2n+1} is $Aut_{CR}(S^{2n+1}) = \mathrm{PU}(n+1,1)$.

2.1.2. We notice that a maximal amenable subgroup of $\mathrm{PU}(n+1,1)$ is isomorphic to the semidirect product $H \bowtie (U(n) \times \mathbb{C}^*)$, where H is the Heisenberg group. $Aut_{CR}(H)$ may be identified with the stabilizer in $\mathrm{PU}(n+1,1)$ of a point in S^{2n+1}. Then $Aut_{CR}(H)$ is a maximal amenable subgroup of $\mathrm{PU}(n+1,1)$ [**BS**].

2.1.3. The nontrivial elements of $\mathrm{PU}(n+1,1)$ fall into three general conjugacy types, depending on the number and location of their fixed points. *Elliptic* elements have a fixed point in $H^{n+1}_{\mathbb{C}}$. *Parabolic* elements have a single fixed point on S^{2n+1}. *Loxodromic* elements have exactly two fixed points on S^{2n+1}. This exausts all possibilities, see [**CG**] for details.

2.1.4 Totally geodesic submanifolds in $H^2_{\mathbb{C}}$. There are two kinds of totally geodesic submanifolds of real dimension 2 in $H^2_{\mathbb{C}}$: *complex geodesics* (represented by $H^1_{\mathbb{C}} \subset H^2_{\mathbb{C}}$) and *totally real geodesic* 2*-planes* (represented by $H^2_{\mathbb{R}} \subset H^2_{\mathbb{C}}$). Each of these totally geodesic submanifold is a model of the hyperbolic plane.

Theorem 2.1. ([**CG**]) *Let M be a totally geodesic submanifold in $H^2_{\mathbb{C}}$ and let $I(M)$ be the stabilizer of M in* $\mathrm{PU}(2,1)$. *Then we have the following.*
i) *If $M = H^1_{\mathbb{C}}$ then $I(M)$ is isomorphic to $U(1) \times \mathrm{PU}(1,1)$,*
ii) *If $M = H^2_{\mathbb{R}}$ then $I(M)$ is isomorphic to* $\mathrm{PSO}(2,1)$.

We will need also the following theorem.

Theorem 2.2. ([**CG**]) *Let G be a subgroup of* $\mathrm{SU}_0(n,1)$*, such that there is no point in $\overline{H}^{n+1}_{\mathbb{C}} = H^{n+1}_{\mathbb{C}} \cup S^{2n+1}$ or proper, totally geodesic submanifold in $H^{n+1}_{\mathbb{C}}$ which is invariant under G. Then G is either discrete or dense in* $\mathrm{SU}_0(n,1)$.

2.2 Uniformization theorems

We recall that a manifold M of dimension $2n+1$ has a spherical CR-structure if M is modeled on the pair $(S^{2n+1}, \mathrm{PU}(n+1,1))$. Therefore, we have the development pair

$$(d^*, d): (Aut_{CR}\widetilde{M}, \widetilde{M}) \to (\mathrm{PU}(n+1,1), S^{2n+1}).$$

2.2.1. Consider an arbitrary subgroup G of $\mathrm{PU}(n+1,1)$. Let $a \in H_{\mathbb{C}}^{n+1}$. The *limit set* of G is defined to be the set $L(G) = \overline{G(a)} \cap S^{2n+1}$. It is easy to see that $L(G)$ does not depend on a.

2.2.2. Let M be a spherical CR-manifold, $p: \widetilde{M} \to M$ the universal covering with deck transformation group $\Gamma = \pi_1(M)$, $d: \widetilde{M} \to S^{2n+1}$ a developing map and $d^*: \Gamma \to \mathrm{PU}(n+1,1)$ a corresponding holonomy homomorphism. Set $D = d(\widetilde{M})$, $\Gamma^* = d(\Gamma)$ and let $N(\Gamma^*) = S^{2n+1} \backslash L(\Gamma^*)$.

Theorem 2.3. (cutting lemma) *Suppose M is a closed manifold with a spherical CR-structure such that $L(\Gamma^*)$ contains more than one point. Let N_0 be the union of the components of $N(\Gamma^*)$ which have a non-empty intersection with D. Let $\widetilde{N_0} = d^{-1}(N_0)$. Then $d_{|\widetilde{N_0}}: \widetilde{N_0} \to N_0$ is a covering map.*

Remark. For closed conformally flat manifolds this theorem was proved by Kulkarni-Pinkall [**KP**]. A slight modification of their arguments gives the proof for spherical CR-structures.

Theorem 2.4. ([**M**]) *Let M be a compact spherical CR-manifold with amenable holonomy group. Then M is finitely covered by the sphere S^{2n+1}, or a Hopf manifold $S^1 \times S^{2n}$, or a compact infranilmanifold.*

Corollary 2.1. *Suppose M is a closed manifold with a spherical CR-structure such that $S^{2n+1} \backslash D$ consists of one or two points. Then $d: \widetilde{M} \to D$ is a homeomorphism, and M is finitely covered by a Hopf manifold or an infranilmanifold.*

Corollary 2.2. *Let M be a compact spherical CR-manifold such that the limit set $L(\Gamma^*)$ is finite. Then $d: \widetilde{M} \to D$ is a homeomorphism and M is finitely covered by S^{2n+1}, or a Hopf manifold, or an infranilmanifold.*

3. Spherical CR-manifolds whose development maps are not surjective

Theorem 3.1. *Let M be a closed 3-dimensional spherical CR-manifold* *with infinite fundamental group. Then the following conditions are equivalent:*

a) $d(\widetilde{M}) = D \neq S^3$,

b) $d\colon \widetilde{M} \to D$ *is a covering map,*

c) *The holonomy group* $\Gamma^* = d^*(\pi_1(M))$ *acts discontinuously on* D.

Proof. Step 1. We will show that a) implies b). Suppose that $S^3 \backslash D$ consists of only one point x_0. Then the holonomy group Γ^* fixes x_0. Applying corollary 2.1 (see section 2.2), we obtain that in this case M is finitely covered by either a Hopf manifold or an infranilmanifold, and d is a homeomorphism.

Now suppose that $S^3 \backslash D$ contains at least two points. Since $S^3 \backslash D$ is closed and invariant under Γ^*, it contains the limit set $L(\Gamma^*)$ of the group Γ^* [**CG**]. It follows from the cutting lemma that in this case $d\colon \widetilde{M} \to D$ is a covering map.

Step 2. We will show that a) implies c). For the reasons explained above, we may assume that $S^3 \backslash D$ contains at least two points. Therefore, if the group Γ^* is discrete, it acts discontinuously on D [**CG**]. Hence, if c) is not satisfied, then Γ^* is not discrete. It follows from theorem 2.2 that we only have the following cases:

i) Γ^* has a fixed point in $H^2_{\mathbb{C}}$,

ii) Γ^* is dense in $\mathrm{PU}(2,1)$,

iii) Γ^* has a fixed point $x_0 \in S^3$,

iv) Γ^* leaves invariant a two point set $\{x_1, x_2\} \subset S^3$,

v) Γ^* leaves invariant some totally geodesic submanifold in $H^2_{\mathbb{C}}$ of real dimension 2.

Note that case i) is impossible, because M would then be modeled by the pair $(S^3, U(2))$. It would then be CR-equivalent to a spherical space form $S^3 \backslash F$, where F is a finite subgroup of $U(2)$, which contradicts our assumption on the fundamental group.

Suppose that case ii) holds. Since $\mathrm{PU}(2,1)$ acts transitively on S^3 and Γ^* is dense in $\mathrm{PU}(2,1)$, it follows that for any two points $a, b \in S^3$, there exists a sequence $\{h_n\} \subset \Gamma^*$ such that $\lim_{n\to\infty} h_n(a) = b$. By taking $a \in S^3 \backslash D$ and $b \in D$, we obtain a contradiction to the openness and invariance of D under Γ^*.

Consider now case iii). Using an appropriate stereographic projection (see section 5.1), we may identify $S^3\backslash\{x_0\}$ with the Heisenberg group H, where x_0 corresponds to ∞. We may suppose that Γ^* contains non-elliptic elements since case i) has already been considered. Thus, there exists an element $h \in \Gamma^*$, h is either loxodromic or parabolic, such that $h(\infty) = \infty$. Suppose that $\infty \in D$. Take a point $a \in S^3\backslash D$. Then $\lim_{n\to\infty} h^{\pm n}(a) = \infty$. It contradicts the openness and invariance of D. Therefore $\infty \in S^3\backslash D$. By applying the arguments in the proof a) $\Rightarrow$ b), we deduce that $d\colon \widetilde{M} \to D$ is a homeomorphism. This implies that Γ^* is discrete and hence, we have arrived to a contradiction.

Suppose that case iv) holds. Then, passing if necessary to a subgroup of index 2 and choosing again a suitable stereographic projection, we may assume that Γ^* contains loxodromic elements since cases i) and iii) have been considered. Thus, there exists a loxodromic element $h \in \Gamma^*$ such that $h(0) = 0$ and $h(\infty) = \infty$. When $\{0, \infty\} \subset D$, we take a point $a \in S^3\backslash D$. Then $\lim_{n\to\infty} h^n(a) \in \{0, \infty\}$, and we have again a contradiction to the openness and invariance of D. Therefore, we may assume that $\infty \in S^3\backslash D$ and achieve a contradiction by applying the arguments in step 1.

The proof of the theorem will be finished in the next section.

3.1 Spherical CR-structures on 3-manifolds with special holonomy

3.1.1. Consider the complex hyperbolic space $H^2_{\mathbb{C}}$ and its boundary $\partial H^2_{\mathbb{C}} = S^3$. We will call $\mathbb{C}$-*circles* the intersections of S^3 with the boundaries of totally geodesic complex submanifolds $H^1_{\mathbb{C}}$ in $H^2_{\mathbb{C}}$. Analogously, we call $\mathbb{R}$-*circles* the intersections of S^3 with the boundaries of totally geodesic real submanifolds $H^2_{\mathbb{R}}$ in $H^2_{\mathbb{C}}$. A subset $K \subset S^3$ will be called a *geometric circle* if K is either a $\mathbb{C}$-circle or a $\mathbb{R}$-circle.

3.2. We will say that a spherical CR-structure on a 3-manifold M is *special* if the holonomy group Γ^* leaves invariant a geometric circle K in S^3.

Theorem 3.2. *Let M be a closed 3-manifold with a special spherical CR-*

structure. Suppose that $\pi_1(M)$ is infinite. Then d is not surjective, and the holonomy group Γ^* is discrete.

Proof. Let K be a geometrical circle invariant under Γ^* and $D = S^3 \backslash K$. We know that $L(\Gamma^*) \subset K$. If $L(\Gamma^*)$ is finite, then by applying corollary 2 in section 2.2, we obtain that $d\colon \widetilde{M} \to S^3 \backslash L(\Gamma^*)$ is a homeomorphism, and, therefore, Γ^* is discrete. Since $\pi_1(M)$ is infinite, $L(\Gamma^*) \neq \varnothing$.

Suppose now that $L(\Gamma^*)$ is infinite. We have two cases to consider.

If $L(\Gamma^*)$ is a proper subset of K, then $S^3 \subset L(\Gamma^*)$ is simply connected. By applying the cutting lemma, we obtain that $d\colon \widetilde{M} \to S^3 \backslash L(\Gamma^*)$ is a homeomorphism, and Γ^* is discrete.

If $L(\Gamma^*) = K$, then it follows from the cutting lemma that

$$d\colon \widetilde{M} \backslash d^{-1}(K) \to S^3 \backslash K$$

is a covering map and thus, d induces a monomorphism

$$d_*\colon \pi_1(\widetilde{M} \backslash d^{-1}(K)) \to \pi_1(S^3 \backslash K) \cong \mathbb{Z}.$$

Suppose that $d(\widetilde{M}) \cap K \neq \varnothing$. Then, using remark 5.5 in [**KP**], we have that $d(\widetilde{M}) = S^3$. A generator of $\pi_1(S^3 \backslash K)$ can be presented by a circle lying in a small neighbourhood of $p \in K$. Since d is a local homeomorphism, it implies that d_* is surjective. It follows that d_* is an isomorphism and, therefore, d must be one to one. A contradiction.

Thus, we have obtained that $d(\widetilde{M}) \cap K = \varnothing$. It is easy to see that in this case $d(\widetilde{M}) = S^3 \backslash K$, and $d\colon \widetilde{M} \to S^3 \backslash K$ is a covering map.

3.2.1. In what follows we suppose that $d(\widetilde{M}) = D$.

3.2.2 Case 1. Suppose that K is a $\mathbb{C}$-circle. Then it follows that $Aut_{CR}D \cong U(1) \times \mathrm{PU}(1,1)$ (see section 2.1.4). Let G be the restriction of $Aut_{CR}D$ to K or, equivalently, to the totally geodesic submanifold in $H^2_{\mathbb{C}}$ with boundary K.

We have the following exact sequence

$$1 \to U(1) \to Aut_{CR}D \xrightarrow{p} G \to 1.$$

The $U(1)$-orbit of any point $a \in D$ is a generator of $\pi_1(D) \cong \mathbb{Z}$. Hence, we get an exact sequence

$$0 \to \mathbb{R} \to Aut_{CR}\widetilde{D} \xrightarrow{\tilde{p}} G \to 1,$$

where $\widetilde{D}$ is the universal covering of D. The exact sequences are related in the following way:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & Aut_{CR}\widetilde{D} & \xrightarrow{\tilde{p}} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & Aut_{CR}D & \xrightarrow{p} & G & \longrightarrow & 1 \end{array} \tag{1}$$

Since $d(\widetilde{M}) = D$ and d is a covering map, we may identify the universal covering $\widetilde{M}$ of the manifold M with $\widetilde{D}$ and $Aut_{CR}\widetilde{D}$. Therefore, we have the development pair

$$(dev^*, dev)\colon (Aut_{CR}\widetilde{D}, \widetilde{D}) \to (Aut_{CR}D, D).$$

Hence, we may think of $\Gamma = \pi_1(M)$ as a subgroup of $Aut_{CR}\widetilde{D}$ and $d^* = dev^*_{|\Gamma}$.

Γ is a discrete cocompact subgroup of $Aut_{CR}\widetilde{D}$. Hence, in particular, the intersection $\mathbb{R} \cap \Gamma$ is cyclic and we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\tilde{p}} & \tilde{p}(\Gamma^*) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H^* & \longrightarrow & \Gamma^* & \xrightarrow{p} & p(\Gamma^*) & \longrightarrow & 1 \end{array} \tag{2}$$

where $H = \mathbb{R} \cap \Gamma$ and $H^* = U(1) \cap \Gamma^*$.

Sometimes H^* will be called the *rotation component* of Γ^*.

Since $U(1)$ acts trivially in $H^1_{\mathbb{C}}$, $g \in \Gamma^*$ and $p(g)$ have the same action in $H^1_{\mathbb{C}}$. In particular, $L(\Gamma^*) = L(p(\Gamma^*))$.

We will show next that $p(\Gamma^*)$ is discrete.

Assume that $p(\Gamma^*)$ is not discrete. Then, by applying theorem 2.2 in section 2.1.4 to $p(\Gamma^*)$, we obtain the following cases:

(i) $p(\Gamma^*)$ has a fixed point $x \in H^1_{\mathbb{C}}$,

(ii) $p(\Gamma^*)$ has a fixed point $x \in K$,

(iii) $p(\Gamma^*)$ leaves invariant some two-point set $\{x_1, x_2\} \subset K$,

(iv) $p(\Gamma^*)$ is dense in G.

As in step 2 of the proof of Theorem 3.1, we obtain that cases (i)-(iii) are impossible.

Consider case (iv). Since $p(\Gamma^*)$ is finitely generated, it follows from corollary 4.5.5 in [**CG**] that $p(\Gamma^*)$ contains elliptic elements of infinite order. Let h_* be an elliptic element of infinite order in $p(\Gamma^*)$ and let $\gamma_* = u_* h_*$ be an element in Γ^* such that $p(\gamma^*) = h_*$, where $u_* \in U(1)$. It is clear that γ_* is elliptic of infinite order and, therefore, $\lim_{n\to\infty} \gamma_*^n = 1$.

Take an element $\gamma \in \Gamma$ such that $d^*(\gamma) = \gamma^*$. Then it follows from diagram (1) that we can compose γ with an element $h \in \mathbb{R}$ to obtain the element $\gamma_1 = h \cdot \gamma$ such that $\lim_{n\to\infty} \gamma_1^n = 1$.

Since every element of $\mathbb{R}$ commutes with Γ, we have $[\gamma, \eta] = [\gamma_1, \eta]$ for all $\eta \in \Gamma$. Therefore, in particular, $[\gamma_1, \eta] \in \Gamma$ for all $\eta \in \Gamma$. As $\lim_{n\to\infty}[\gamma_1^n, \eta] = 1$ and Γ is discrete, we obtain that $[\gamma_1, \eta] = 1$ for all $\eta \in \Gamma$. It follows that γ_* commutes with every element of Γ^*. Since $h_* \neq 1$, it follows that $p(\Gamma^*)$ must be abelian and, hence, we have again cases (i)-(iii) above, which, as shown, are impossible.

Thus, we have shown that $p(\Gamma^*)$ is discrete.

Suppose now that Γ^* is not discrete. Since $p(\Gamma^*)$ is discrete, it follows from diagram (2) that Γ^* is not discrete if and only if $\mathrm{Ker}(d^*\colon \Gamma \to \Gamma^*)$ is trivial. In this case, the rotation component $H^* \cong \mathbb{Z}$.

As Γ and Γ^* are finitely generated, by passing to subgroups of finite index, we may assume that Γ^* is torsion-free.

Thus, we have that $p(\Gamma^*)$ is discrete, torsion-free, non-solvable, finitely generated subgroup of G. Then we know that $p(\Gamma^*)$ is either a finitely generated non-abelian free group or isomorphic to the fundamental group of a closed surface of genus ≥ 2.

Suppose that $p(\Gamma^*)$ is a free group. Then $L(p(\Gamma^*)) = L(\Gamma^*)$ is a Cantor set lying in K. This contradicts the fact that $L(\Gamma^*) = K$.

The final claim is that $p(\Gamma^*)$ cannot be isomorphic to the fundamental group of a closed surface of genus ≥ 2.

Assume that $p(\Gamma^*)$ is isomorphic to the fundamental group of a closed surface of genus ≥ 2. Then it follows from diagram (2) that the manifold M is homeomorphic to a circle bundle over a closed hyperbolic surface [**S**]. On the other hand, under our hypothesis, M is modeled on the pair $(Aut_{CR}D, D)$ and, therefore, as a circle bundle, it has nonzero Euler number [**BS, G3**]. It is well known that in this case Γ has no subgroups isomorphic to the fundamental group of a closed surface of genus ≥ 2 [**S**]. Since $d^*: \Gamma \to \Gamma^*$ is an isomorphism, diagram (2) shows again that we have arrived to a contradiction.

Thus, we have proved that in case 1 the holonomy group is discrete.

3.2.3 Case 2. Suppose now that K is a $\mathbb{R}$-circle. Then it follows from theorem 2.1 in section 2.1.4 that $Aut_{CR}D$ is the image of the imbedding $\mathrm{SO}(2,1) \to \mathrm{PU}(2,1)$ obtained by composing the imbedding $\mathrm{SO}(2,1) \to U(2,1)$ with projectivization.

Let $\widetilde{D}$ be a universal covering of D. As in Case 1, since $d(\widetilde{M}) = D$, we may identify a universal covering $\widetilde{M}$ of the manifold M with $\widetilde{D}$ and $Aut_{CR}\widetilde{M}$ with $Aut_{CR}\widetilde{D}$.

The following has been obtained in [**BS**].

Proposition 3.1. *D is isomorphic to the unit tangent circle bundle of the two dimensional real hyperbolic space $H^2_{\mathbb{R}}$.*

Therefore, we have that

$$(Aut_{CR}D, D) = (\mathrm{PSO}(2,1), T_1 H^2_{\mathbb{R}})$$

and the result we need follows from theorem 7.2 in [**KR**].

3.3 End of the proof of theorem 3.1

3.3.1. Suppose now that case v) in step 2 occurs. Note, first of all, that $L(\Gamma^*) \subset K$. For, if $L(\Gamma^*) = K$, we obtain that $d(\widetilde{M}) = S^3 \backslash K$ and we can finish the proof applying Theorem 3.2. If $L(\Gamma^*)$ is a proper subset of K then $S^3 \backslash L(\Gamma^*)$ is simply-connected, and we are in the situations of step 1. The proof there shows that Γ^* is discrete.

3.3.2 Step 3. b) implies a) and c) implies a). We note first that the implication b) $\Rightarrow$ a) is trivial, since $\widetilde{M}$ is noncompact, while S^3 is compact and simply-connected.

Let us show that c) $\Rightarrow$ a). If $d(\widetilde{M}) = S^3$, then since Γ^* acts discontinuously and S^3 is compact, Γ^* is a finite group. In this case Γ^* is purely elliptic and consequently is a subgroup of the unitary group $U(2)$. Thus, M is modeled on the pair $(S^3, U(2))$. Since M is close, it implies that its fundamental group is finite, which contradicts the hypothesis of the theorem.

3.3.3. One sees that we have proved that a $\Leftrightarrow$ b and a $\Leftrightarrow$ c. Thus, the theorem is proved.

4. Spherical CR-structures on S^1-bundles over surfaces

4.1 Standard spherical CR-structures on S^1-bundles over surfaces

Let H_g denote a group isomorphic to the fundamental group of a closed orientable surface S_g of genus $g \geq 2$. Suppose that $\rho: H_g \to P(U(2,1))$ is a homomorphism. We say that ρ is a *discrete embedding* if and only if ρ is injective and its image $\rho(H_g)$ is a discrete subgroup of $\mathrm{PU}(2,1)$.

There are two special kinds of discrete embeddings of H_g into $\mathrm{PU}(2,1)$.

4.1.1. Let $H^1_{\mathbb{C}}$ be a totally geodesic complex submanifold in $H^2_{\mathbb{C}}$. We will consider $H^1_{\mathbb{C}}$ as the set $\{(z_1, z_2) \in B^2 : z_1 = 0\}$.

Assume now that H_g is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$ and suppose that H_g is generated by $\gamma_1, \ldots, \gamma_{2g}$, with

$$\gamma_i = \begin{pmatrix} a_i & b_1 \\ c_i & d_i \end{pmatrix}$$

Let H_g act on $H^2_{\mathbb{C}}$ by

$$\gamma_i(z_1, z_2) = \left(\frac{z_1}{c_i z_2 + d_i}, \frac{a_i z_2 + b_i}{c_i z_2 + d_i} \right).$$

This action corresponds to the standard imbedding of $\mathrm{SU}(1,1)$ into $\mathrm{PU}(2,1)$ given by composing the embedding $U(1,1) \to U(2,1)$

$$A \to \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$$

with projectivization $U(2,1) \to \mathrm{PU}(2,1)$. We denote $H_{\mathbb{C}}$ the image of H_g corresponding to this embedding.

Let $D_{\mathbb{C}} = S^3 \backslash K_{\mathbb{C}}$, where $K_{\mathbb{C}} = \partial H^1_{\mathbb{C}}$. Then $H_{\mathbb{C}}$ acts discontinuously on $D_{\mathbb{C}}$, and the limit set of $H_{\mathbb{C}}$ equals $K_{\mathbb{C}}$. It is well known [**BS, G3**] that the manifold $M(H_{\mathbb{C}}) = D_{\mathbb{C}}/H_{\mathbb{C}}$ is homeomorphic to the circle bundle over S_g whose Euler number is $1 - g$.

Thus, we have that any S^1-bundle over S_g with Euler number $e = 1 - g$ admits a uniformizable spherical CR-structure.

4.1.2. Let $H^2_{\mathbb{R}}$ be a totally geodesic real submanifold in $H^2_{\mathbb{C}}$.As the identity component $\mathrm{SO}(2,1)^0 \cong \mathrm{PSL}(2,R)$, there exists a discrete embedding H_g into $\mathrm{PU}(2,1)$ given by

$$H_g \to \mathrm{SO}(2,1) \subset U(2,1) \to \mathrm{PU}(2,1)$$

We denote $H_{\mathbb{R}}$ the image of H_g corresponding to this embedding.

Let $D_{\mathbb{R}} = S^3 \backslash K_{\mathbb{R}}$, where $K_{\mathbb{R}} = \partial H^1_{\mathbb{R}}$. Then $H_{\mathbb{R}}$ acts discontinuously on $D_{\mathbb{R}}$, and the limit set of $H_{\mathbb{R}}$ equals $K_{\mathbb{R}}$. The manifold $M(H_{\mathbb{R}}) = D_{\mathbb{R}}/H_{\mathbb{R}}$ is homeomorphic to a circle bundle over S_g whose Euler number is $2g - 2$ [**BS**], [**KR**].

Thus, we have that any circle bundle over S_g with Euler number $e = 2g - 2$ admits a uniformizable spherical CR-structure.

4.1.3. The spherical CR-structures on S^1-bundles constructed above will be called *standard*.

4.1.4. Note that in both cases above the spherical CR-structures on S^1-bundles over S_g are special and their holonomy groups coincide with discrete embeddings H_g constructed in 4.1.1 and 4.1.2.

4.2 Non-standard spherical CR-structures on $\mathbf{S^1}$-bundles over surfaces

In this section we will construct special spherical CR-structures on S^1-bundles over closed orientable surfaces of genus $g \geq 2$ with arbitrary Euler numbers $e \neq 0$.

4.2.1. Before describing the constructions, we establish the following notations. $E(g,e)$ will denote a circle bundle over a closed orientable surface of genus $g \geq 2$ with Euler number e, Γ will denote the group of deck transformations of the universal covering space of $E(g,e)$. Recall that Γ has a presentation:

$$\Gamma = \langle a_1, b_1, \dots, a_g, b_g, h \colon \prod_{i=1}^{i=g} [a_i, b_i] = h^e, h \text{ central} \rangle$$

4.2.2. Consider the standard spherical CR-structure on $E(g,e)$ constructed in 4.1.1. Let $d^*\colon \Gamma \to \Gamma^* \subset \mathrm{PU}(2,1)$ be the corresponding holonomy homomorphism, Γ^* be the holonomy group. Then d^* has a cyclic kernel, generated by h, and

$$\Gamma^* = d(\Gamma^*) \cong \Gamma/\langle h \rangle = \langle a_1, b_1, \dots, a_g, b_g \colon \prod_{i=1}^{i=g} [a_i, b_i] = 1 \rangle = H_{\mathbb{C}}.$$

Diagram (2) in this case becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\tilde{p}} & \tilde{p}(\Gamma) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \Gamma^* & \xrightarrow{p} & \Gamma^* & \longrightarrow & 1 \end{array}$$

Let U_k be a cyclic subgroup of $U(1)$ of order $k \geq 1$. Consider the group $\Gamma_k^* = \langle \Gamma^*, U_k \rangle$ generated by Γ^* and U_k. It is clear that Γ_k^* is a discrete subgroup of $\mathrm{PU}(2,1)$, Γ_k^* acts discontinuously on $D_{\mathbb{C}}$, the limit set $L(\Gamma_k^*) = K_{\mathbb{C}}$, Γ_k^* is the direct product of U_k and Γ^*, Γ^* is a subgroup of Γ_k^* of index k, Γ_k^* acts without fixed points on $D_{\mathbb{C}}$.

Next note that Γ_k^* is the holonomy group of the spherical CR-manifold $M_k = D_{\mathbb{C}}/\Gamma_k^*$ which is uniformizable by Γ_k^* (see section 4.3).

Diagram (2) in this case becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & \Gamma_k & \xrightarrow{\tilde{p}} & \tilde{p}(\Gamma_k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U_k & \longrightarrow & \Gamma_k^* & \xrightarrow{p} & \Gamma^* & \longrightarrow & 1 \end{array}$$

Let $M_1 = D_{\mathbb{C}}/\Gamma^*$. As was shown above, $M_1 \cong E(g, 1-g)$.

Consider the covering $M_1 \to M_k$ induced by the inclusion $\Gamma^* \subset \Gamma_k^*$. Then it is easy to see that the manifold M_1 is a k-fold cyclic covering of M_k obtained by dividing M_1 by the action of the cyclic subgroup of order k in S^1. Therefore, M_k is a S^1-bundle over S_g. It follows from lemma 3.5 in [**S**] that the Euler number of M_k equals $k(1-g)$. Thus, $M_k \cong E(g, k(1-g))$. Since $1-g \neq 0$, for any $k > 1$ $E(g,k)$ is a $(g-1)$-fold covering of M_k.

Lifting the spherical CR-structure on M_k to $E(g,k)$ and noting that $E(g,k)$ is homeomorphic to $E(g,-k)$, we obtain the following theorem.

Theorem 4.1. *Let $E(g,e)$ be a circle bundle over a closed orientable surface of genus $g \geq 2$ with non zero Euler number e. Then $E(g,e)$ admits a spherical CR-structure.*

Remark. In general, the spherical CR-structure on $E(g,e)$ constructed above is not uniformizable. We will discuss this in the next section.

4.3 Kleinian and non-Kleinian structures

The most natural class of (X,G)-structures arises as follows. Let D be an open connected subset of X and Γ be a subgroup of G which leaves D invariant and where it acts freely and discontinuously. Then the manifold $M = D/\Gamma$ clearly admits a natural (X,G)-structure. We will call such a structure on M *Kleinian* or *uniformizable.* More generally, an (X,G)-structure on a manifold M will be called uniformizable or Kleinian if it is (X,G)-equivalent to a Kleinian structure defined as above. Two (X,G)-structures on M are called *commensurable* if they have (X,G)-equivalent finite coverings. We will call an (X,G)-structure on M *virtually uniformizable* or *virtually Kleinian* if it is commensurable to a Kleinian structure. Finally, we will call an (X,G)-structure on M *almost Kleinian* if $d\colon \widetilde{M} \to d(\widetilde{M})$ is a covering map.

A problem of basic geometric interest is to find criteria for an (X,G)-structure to be Kleinian, virtually Kleinian or almost Kleinian. For the case of conformally flat structures this problem was considered in [**G1**], [**K**], [**GKam1**], [**GKam2**], [**KP**], [**GK**]. Theorems 3.1 and 3.2 are the first step in this direction for spherical CR-structures.

4.3.1. The standard spherical CR-structures on S^1-bundles over closed orientable surfaces S_g of genus $g \geq 2$ constructed in 4.1 provide examples of Kleinian structures.

4.3.2. Now we present examples of virtually Kleinian but non-Kleinian spherical CR-structures on S^1-bundles over S_g.

Example 1. Let $M^1 = E(g, 1-g)$ be the S^1-bundle over S_g equipped with the standard spherical CR-structure constructed in 4.1.1. Then the holonomy group of this structure is $\Gamma^* = d^*(\Gamma) \cong \pi_1(S_g)$.

Take $g-1 = kn$, where k and n are positive integers greater then 1. Consider the k-fold covering $p_k: M^k \to M^1$ with the defining subgroup $\Gamma^k \subset \Gamma$,

$$\Gamma^k = \langle a_1, b_1, \dots, a_g, b_g, h^k \rangle.$$

Then $M^k \cong E(g, -n)$.

Define the spherical CR-structure on M^k by lifting the spherical CR-structure on M^1. Let $d_k: \widetilde{M} \to D_{\mathbb{C}}$ be the corresponding development map. It follows from the construction that $\operatorname{Ker} d_k^* = \operatorname{Ker} d^* \cap \Gamma^k = \langle h^k \rangle$. Hence, the holonomy group $\Gamma_k^* = d_k^*(\Gamma^*)$ of this spherical CR-structure on M^1 coincides with Γ^*. We see that the spherical CR-manifold M^1 is uniformized by its holonomy group Γ^*, while M^k is not. Thus, M^k provides an example of virtually Kleinian structure which is not Kleinian.

Remark. It is easy to see that the same arguments work for the spherical CR-manifolds constructed in 4.1.2.

Example 2. Here we consider in more detail the spherical CR-manifolds constructed in the proof of Theorem 4.2.

Let $\Gamma_k^* = \langle \Gamma^*, U_k \rangle$, where Γ^* is the group constructed in 4.1.1, and $U_k \subset U(1)$ is a finite cyclic group of order $k \geq 1$. Then as was shown in section 4.2.2, the quotient $M_k = D_{\mathbb{C}}/\Gamma_k^*$ is a spherical CR-manifold homeomorphic to $E(g, k(1-g))$, and Γ_k^* is its holonomy group, that is, M_k is uniformizable.

Suppose that k and $g-1$ are both primitive integers. Then there are only three non-trivial finite covers of M_k which are S^1-bundles over S_g:

i) $p_k: E(g, 1-g) \to M_k$,

ii) $p_{g-1}: E(g, -k) \to M_k$,

iii) $p_{k(g-1)}: E(g, -1) \to M_k$.

Define the spherical CR-structure on these manifolds by lifting the spherical CR-structure on M_k. Then in cases i) and ii) the structures are Kleinian and their holonomy groups are subgroups of Γ_k^* of index k

and $g-1$ respectively. In case iii) the structure is not Kleinian and its holonomy group coincides with Γ_k^*.

Example 3. Let $M \cong E(g, 2g-2)$ be the S^1-bundle over S_g equipped with the standard spherical CR-structure constructed in 4.1.2. We know that

$$\Gamma = \pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, h\colon \prod_{i=1}^{g} [a_i, b_i] = h^{2g-2}, h \text{ central}\rangle.$$

Consider the 2-fold covering $p_2\colon M_2 \to M$ corresponding to the subgroup $\Gamma_2 \subset \Gamma$,

$$\Gamma_2 = \langle a_1, b_1, \ldots, a_g, b_g, h^2\rangle.$$

Then $M_2 \cong E(g, g-1)$. Define a spherical CR-structure on M_2 using p_2.

We have that $E(g, g-1)$ is homeomorphic to $E(g, 1-g)$, so M_2 can be equipped with a standard spherical CR-structure as in 4.1.1.

Thus, we see that the manifold M_2 admits two spherical CR-structures: one of them is Kleinian, while the other is not. Of course, these structures are not CR-equivalent.

4.3.3. In this section we present an example of a spherical CR-manifold N with infinite fundamental group which has a surjective development map, that is, a spherical CR-structure on the manifold N which is not almost Kleinian.

Let M^k be the spherical CR-manifold constructed in example 1 in 4.3.2, where $k > 1$.

Let $d_k\colon \widetilde{M} \to D_{\mathbb{C}}$ be the development map and $p\colon \widetilde{M} \to M^k$ be the universal cover of M.

Let $B \subset M^k$ be a small open ball in M^k. It is easy to see that $d_k(p^{-1}(M^k \setminus B)) = d_k(\widetilde{M}) = D_{\mathbb{C}}$.

Take any closed 3-dimensional spherical CR-manifold M with infinite fundamental group. Let B' be a small open ball in M. We may define a spherical CR-structure on the connected sum $N = M^k \sharp M$ along the boundaries B and B' using the construction in [**BS**], [**F**].

Let $d\colon \widetilde{N} \to S^3$ be the development map of this spherical CR-structure. Then it follows from the above that $d(\widetilde{N}) \cap L(M) \neq \varnothing$, where $L(M)$ is the limit set of the holonomy group of the spherical CR-structure on M. Since the limit set $L(N)$ of the holonomy group of N is obviously infinite, it follows from Remark 5.5 in [**KP**] that $d(\widetilde{N}) = S^3$.

5. Fundamental domains

In this section we give explicit constructions of fundamental domains for some discrete subgroups of the group of conformal transformations of the one-point compactification $\overline{H}$ of the Heisenberg group. For a notion of conformality on $\overline{H}$ the reader is referred to Koranyi and Riemann [**KoR**].

5.1 The stereographic projection and the Heisenberg group

The mapping

$$C\colon \begin{cases} z_1 &= \frac{iw_1}{1+w_2} \\ z_2 &= \frac{1-w_2}{1+w_2} \end{cases}$$

is usually referred to as the *Cayley transform.* The Cayley transform maps the unit ball

$$B = \{w \in \mathbb{C}^2 \colon |w_1|^2 + |w_2|^2 < 1\}$$

biholomorphically onto

$$V = \{z \in \mathbb{C}^2 \colon \operatorname{Im} z_2 > |z_1|^2\}$$

The Cayley transform leads to a generalized form of the *stereographic projection.* This mapping $\pi\colon S^3 \backslash \{-e_2\} \to \mathbb{R}^3$, where $S^3 = \partial B$ and $e_2 = (1,0) \in \mathbb{C}^2$, is defined as the composition of the Cayley transform restricted to $S^3 \backslash \{-e_2\}$ followed by the projection

$$\begin{cases} z_1 & \to z_1 \\ z_2 & \to \operatorname{Re} z_2 \end{cases}$$

The stereographic projection π can be extended to a mapping from S^3 onto the one-point compactification $\overline{\mathbb{R}}^3$ of $\mathbb{R}^3$.

The *Heisenberg group* H is the set of pairs $[t,z] \in \mathbb{R} \times \mathbb{C}$ with the product

$$[t,z] \cdot [t',z'] = [t + t' + 2\operatorname{Im}(z\overline{z}'), z + z']$$

Using the stereographic projection, we can identify $S^3\backslash\{-e_2\}$ with H and S^3 with the one-point compactification $\overline{H}$ of H.

5.2. The Heisenberg group acts on itself by left translations. Heisenberg translations by $[0, v]$ for $v \in \mathbb{R}$ are called *vertical translations.*

Positive scalars $\lambda \in \mathbb{R}_+$ act on H by *Heisenberg dilations*:

$$d_\lambda: [t, z] \to [\lambda^2 t, \lambda z].$$

If $m \in U(1)$, then m acts on H by

$$m: [t, z] \to [t, mz],$$

m is called a *Heisenberg rotation.*

The *Heisenberg inversion* of H is defined on $H\backslash\{\text{origin}\}$ by

$$h: [t, z] \to \left[-\frac{t}{t^2 + |z|^4}, \frac{z}{it - |z|^2}\right].$$

Note that $h = \pi \circ j \circ \pi^{-1}$, where j is the involution

$$j: \begin{cases} w_1' &= -w_1 \\ w_2' &= -w_2 \end{cases} \quad (w_1, w_2) \in \mathbb{C}^2.$$

The map $\widehat{m}$ defined by

$$\widehat{m}: [t, z] \to [-t, \overline{z}].$$

All these actions extend trivially to $\overline{H}$. It is well known that the group G of transformations of $\overline{H}$ generated by all Heisenberg translations, dilations, rotations, and h coincides with $\pi^{-1} \circ \mathrm{PU}(2,1) \circ \pi$, and the group $\widehat{G} = \langle G, \widehat{m} \rangle$ is the group of all conformal transformations of $\overline{H}$ [**KoR**].

5.3. *The Heisenberg norm* assigns to $g = [t, z]$ in H the nonnegative real number

$$|g| = (|z|^4 + t^2)^{1/4}.$$

The function $d(g, g') = \left|g^{-1} g'\right|$ defines a distance on H. Heisenberg translations and rotations are isometries with respect to this distance. Furthermore, $|d_\lambda g| = \lambda_{|g|}$ and $|\widehat{m} g| = |g|$.

5.4. We will call the *Heisenberg sphere* (H-sphere) with center a and radius ρ the set

$$S(a,\rho) = \{g \in H : d(a,g) = \rho\}.$$

5.5. Let S be the H-sphere with center at the origin and radius 1. It is easy to see that $h(S) = S$, $h(\operatorname{ext} S) = \operatorname{int} S$, where $\operatorname{int} S = \{g \in H : d(0,g) < 1\}$, $\operatorname{ext} S = \{g \in H : d(0,g) > 1\}$. Thus, we see that h has some features of the usual euclidean inversions in spheres, and it is natural to call h the *inversion* in the H-sphere S.

Example 1. Let $\Gamma = \langle h \rangle$ be the group generated by h. Then it follows from above that $F = \operatorname{int} S$ is a fundamental domain for Γ.

5.6. It is useful to consider the following transformation

$$I = \widehat{m} \circ h : [t,z] \to \left[\frac{t}{t^2 + |z|^4}, \frac{-\bar{z}}{it + |z|^2}\right].$$

Observe that I leaves invariant S as well as the circles $|z|^2 = \sqrt{1-t}$, $|t| < 1$, and $I(\operatorname{int} S) = \operatorname{ext} S$.

Define $I_g = g \circ I \circ g^{-1}$, where g is either a Heisenberg translation or a Heisenberg dilation. It is easy to see that the H-sphere $S_g = g(S)$ is invariant under I_g and $I_g(\operatorname{int} S_g) = \operatorname{ext} S_g$. We will also call I_g the inversion in S_g.

5.7 Example 2. Let S_1 and S_2 be the H-spheres of radius 1 centered at the points

$$o_1 = \left[-\frac{\sqrt{2}}{2}, 0\right] \quad \text{and} \quad o_2 = \left[\frac{\sqrt{2}}{2}, 0\right]$$

respectively. Consider the inversions $\gamma_1 = I_{g_1}$ and $\gamma_2 = I_{g_2}$ in S_1 and S_2, where

$$g_1 = \left[\frac{\sqrt{2}}{2}, 0\right] \quad \text{and} \quad g_2 = \left[-\frac{\sqrt{2}}{2}, 0\right]$$

are the vertical translations. A simple calculation shows that γ_i leaves invariant $S_j, i \neq j$, furthermore, γ_i leaves invariant the circle $c = S_1 \cap S_2$.

Let $\Gamma = \langle \gamma_1, \gamma_2 \rangle$. A direct verification gives that $F = \operatorname{ext}(S_1) \cap \operatorname{ext}(S_2)$ is a fundamental domain for Γ.

We also note the presentation of Γ:

$$\Gamma = \langle \gamma_1, \gamma_2 \colon \gamma_1^2 = \gamma_2^2 = (\gamma_1 \circ \gamma_2)^2 = 1 \rangle.$$

5.8 Example 3. Let $S(0, \lambda)$ be the H-sphere with radius λ centered at the origin, that is, $S(0, \lambda) = d_\lambda(S)$, where d_λ is a Heisenberg dilation; and let $S(h, 1)$ be the H-sphere with radius 1 centered at the point $[h, 0]$, that is, $S(h, 1) = t_h(S)$, where $t_h = [h, 0]$ is a vertical translation.

Consider now the corresponding inversions

$$I_\lambda = d_\lambda \circ I \circ d_\lambda^{-1} \quad \text{and} \quad I_h = t_h \circ I \circ t_h^{-1}$$

in $S(0, \lambda)$ and $S(h, 1)$ respectively.

Calculations show that $S(0, \lambda)$ is invariant under I_h if and only if $\lambda^4 = h^2 - 1$. On the other hand, it is also seen that under this condition on λ and h, $S(h, 1)$ is invariant under I_λ. Furthermore, the circle $S(0, \lambda) \cap S(h, 1)$ is invariant under both I_h and I_λ.

Also it is easy to see that $S(j\sqrt{2}, 1)$ is invariant under both $I_{(j-1)\sqrt{2}}$ and $I_{(j+1)\sqrt{2}}$, $j \in \mathbb{Z}$. The circle $S((j-1)\sqrt{2}, \lambda) \cap S(j\sqrt{2}, 1)$ is invariant under both $I_{(j-1)\sqrt{2}}$ and $I_{j\sqrt{2}}$.

For each integer $n \geq 2$, consider the following family L of H-spheres:

$$S_0 = S(0, 1), S_j = S(h_j, 1),$$
$$S_0' = S(0, \lambda), S_j' = S(-h_j, 1),$$

where $1 \leq j \leq n$, $h_j = \sqrt{2}j$ and $\lambda = (2n^2 - 1)^{1/4}$.

Next define the transformations $\gamma_0, \gamma_0', \gamma_1, \gamma_1', \ldots, \gamma_n, \gamma_n'$ as follows

$$\gamma_0 = I, \gamma_0' = I_\lambda, \gamma_j = I_{h_j}, \gamma_j' = I_{-h_j}$$

for $1 \leq j \leq n$.

Let $\Gamma(n) = \langle \gamma_0, \gamma_0', \gamma_1, \gamma_1', \ldots, \gamma_n, \gamma_n' \rangle$ be the group generated by the transformations defined above.

It is clear that Γ leaves invariant $D = H \backslash \{t\text{-axis }\}$.

Now let P be the "spherical polyhedron" bounded by the H-spheres S_j, S_j' for $0 \leq j \leq n$, that is, $P = \overline{F}$, where

$$F = \operatorname{ext} S_0 \cap \operatorname{int} S_0' \cap (\cap_{j=1}^n \operatorname{ext} S_j) \cap (\cap_{j=1}^n \operatorname{ext} S_j').$$

We call an edge of P the circle c which is the intersection of two spheres in L. A part of the boundary of P lying on $S_j \in L$ between two edges will be called the side of P.

It follows from the construction that we have the following:

i) P is compact,

ii) For each side A of P there exists a transformation $\gamma_A \in \{\gamma_j, \gamma_j'\}$ such that $P \cap \gamma_A(P) = A$,

iii) For each side A of P there exists a side A' such that $\gamma_A \circ \gamma_{A'} = 1$ (of course, $A = A'$ and $\gamma_A = \gamma_{A'}$),

iv) For each edge c of P there exists a sequence $A_1, \dots, A_k$ of sides of P such that $\gamma_{A_1} \circ \dots \circ \gamma_{A_k} = 1$ and

$$P \cap \gamma_{A_1}(P) \cap \gamma_{A_1} \circ \gamma_{A_2}(P) \cap \dots \cap \gamma_{A_1} \cdots \gamma_{A_{k-1}}(P) = c,$$

v) The polyhedra $P, \gamma_{A_1}(P), \dots, \gamma_{A_1} \circ \dots \circ \gamma_{A_{k-1}}(P)$ do not have pairwise common interior points.

We know that the t-axis completed by ∞ is the image of a $\mathbb{C}$-circle in S^3; it corresponds under the stereographic projection π to the set $\{(w_1, w_2) \in S^3, w_2 = 0\}$. Therefore, one can introduce a complete Riemannian metric on $D = H \backslash \{t\text{-axis}\}$ invariant under the group Γ (see, for instance, [**KT**]).

Applying similar arguments to those in the proof of the Poincaré's Polyhedron theorem [**Ma**], we conclude that the construction above yields a fundamental domain F for Γ. Furthermore, the limit set $L(\Gamma)$ of Γ equals the t-axis completed by ∞.

Since Γ is finitely generated, there exists a torsion-free subgroup Γ_0 of finite index in Γ. Then $M(\Gamma_0) = D \backslash \Gamma_0$ is a circle bundle over a closed hyperbolic surface with non-zero Euler number (see section 4).

5.9 Klein's combination theorem

Theorem 5.1. *Let Γ_1 and Γ_2 be discrete subgroups of* $\mathrm{PU}(2,1)$ *with fundamental domains F_1 and F_2. Suppose that $F_1 \cup F_2 = S^3$ and $F = F_1 \cap F_2$ is connected and non-empty. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is discrete with fundamental domain F.*

Example 4. Consider the following conformal transformation of B:

$$(z_1, z_2) \to (z_2, z_1).$$

The action of this transformation on H under the stereographic projection π corresponds to

$$\hat{h}\colon [t, z] \to \left[\operatorname{Re} i\frac{1+|z|^2 - it + 2iz}{1+|z|^2 - it - 2iz}, i\frac{1-|z|^2+it}{1+|z|^2-it-2iz}\right].$$

Let $\Gamma_1 = \Gamma(n_0)$ be the group constructed in example 3 for any fixed $n_0 > 1$, and F_1 be its fundamental domain. Let $\Gamma_2 = \hat{h} \circ \Gamma_1 \circ \hat{h}^{-1}$ and $F_2 = \hat{h}(F_1)$. We see that the limit set $L(\Gamma_2)$ of the group Γ_2 is the unit circle centered at the origin

$$L(\Gamma_2) = \{g = [t, z] \colon ||g|| = 1, t = 0\}.$$

The boundary of F_2 is the boundary of the solid torus having $L(\Gamma_2)$ as its core. It is clear that there exists a Heisenberg dilation d_s such that $d_s(\partial F_2)$ lies in the complement of all the balls bounded by $S_j, j = 0, \dots, n$, and $S'_k, k = 1, \dots, n$. Having defined such s, choose n_1 such that $d_s(\partial F_2) \subset \operatorname{int} S(0, \lambda_1)$, where $\lambda_1 = (2n_1^2 - 1)^{1/4}$. Let $\Gamma'_1 = \Gamma(n_1)$ be the group constructed in example 3 corresponding to $n = n_1$ and F'_1 be its fundamental domain. Let $\Gamma'_2 = d_s \Gamma_2 d_s^{-1}$. Then $F'_2 = d_s(F_2)$ is a fundamental domain for Γ'_2. One sees that the complement $(F'_i) \subset F'_j, i, j = 1, 2, i \neq j$. It follows that the conditions of Klein's combination theorem are satisfied and, therefore, $\Gamma = \langle \Gamma'_1, \Gamma'_2 \rangle$ is discrete with the fundamental domain $F = F'_1 \cap F'_2$.

The limit set $L(\Gamma)$ is quite complicated. In particular, it contains the Γ-orbit of $\{T \cup d_s \hat{d}(T)\}$, where T is the t-axis completed with ∞.

If Γ_0 is a subgroup of finite index in Γ without torsion, then $M(\Gamma_0) = R(\Gamma)/\Gamma_0$ is an aspheric manifold. Here, $R(\Gamma) = \overline{H} \backslash L(\Gamma)$ is the regular set of Γ.

One can show that $M(\Gamma_0)$ is a torus sum of S^1-bundles over a compact hyperbolic surface.

$M(\Gamma_0)$ provides the first example of an aspheric manifold with a spherical CR-structure which is not a Seifert manifold.

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